Tiling-Harmonic Functions

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Background

- We are studying functions on the vertices of square tilings.
- A square tiling is defined broadly as a connected set of squares in the plane with disjoint interiors and whose edges are parallel to the coordinate axes.
- This project works with subsets of the regular square lattice $\mathbb{Z}^2$.

- A tiling S is a subtiling of a tiling T if the set of its squares is a subset of the set of T's.

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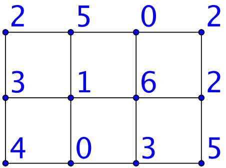

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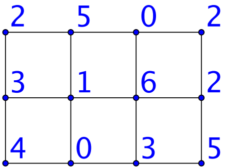

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Definition

The oscillation $osc(u,t)$ of a function on a square t is the difference between the maximum and minimum values on that square.

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The energy $E(u)$ of a function on a tiling is the sum over all squares t in that tiling of $(osc(u,t))^2$.

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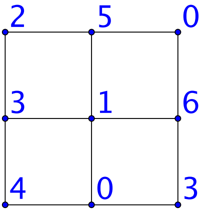
Calculating the Energy

Example



gives the energy

$$E(u) = (5-1)^2 + (6-0)^2 + (4-0)^2 + (6-0)^2 = 104.$$

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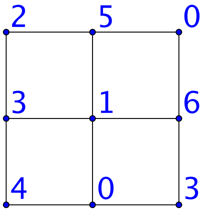

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- A function on a finite square tiling is called tiling-harmonic if its energy is minimized among all functions on that tiling with the same boundary values.
- A function on an infinite tiling is tiling-harmonic if it is tiling-harmonic on all finite subtilings.

Remark

Given a tiling and a set of boundary values, tiling-harmonic functions are not necessarily unique.

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A Tiling-Harmonic Function

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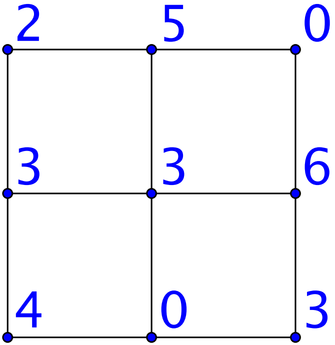
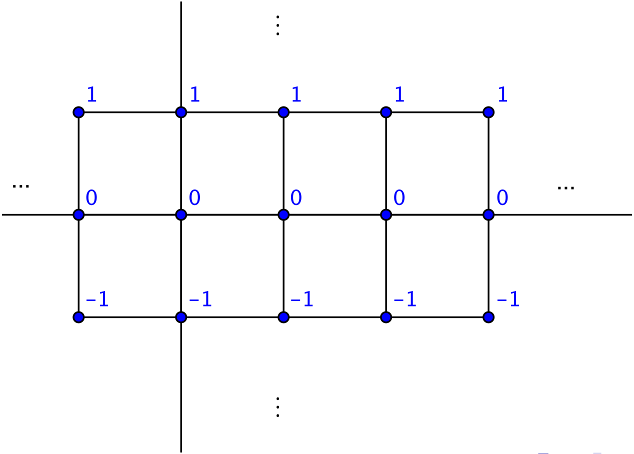


Another Tiling-Harmonic Function

Theorem

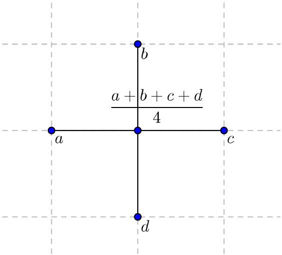
The function $f(x, y) = y$ is tiling-harmonic.



Graph Harmonic Functions

Definition

A function on a square tiling is called graph-harmonic if the value at each vertex is the average of the values of its neighbors.



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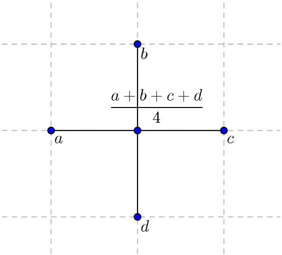

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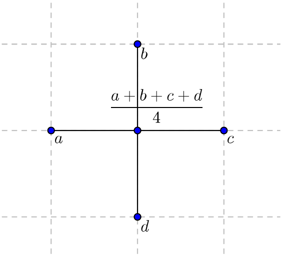

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Two Main Conjectures

Conjecture (Liouville's Theorem for TH Functions)

A bounded tiling-harmonic function on the regular lattice grid ($\mathbb{Z}^2$) must be constant.

- Liouville's is a major theorem for harmonic functions.
- This theorem serves as a "simpler version" of the second, more important conjecture.

Conjecture

A tiling-harmonic function on the upper half-plane that vanishes along the x-axis must be proportional to y.

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- The second conjecture may provide an alternative proof of the quasisymmetric rigidity of square Sierpinski carpets.
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Results — Harnack's Inequality

Conjecture (Harnack's Inequality)

On a nonnegative tiling-harmonic function, the ratio of the values on two points a fixed distance r apart is bounded.

- Proving Harnack's Inequality would be a major step toward proving Liouville's Theorem.
- Harnack's Inequality is known for graph harmonic functions.
- We have strong experimental evidence that the maximum ratio for a tiling-harmonic function is bounded by that of the graph harmonic function with the same boundary values.
- A proof of this bound would imply Harnack's Inequality for tiling-harmonic functions.

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Results — Maximum Modulus Principle

Theorem (Maximum Modulus Principle)

On an m x n rectangular grid with $m, n \geq 4$, if the maximum value occurs on the interior, then the entire set of interior values is constant.

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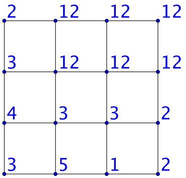

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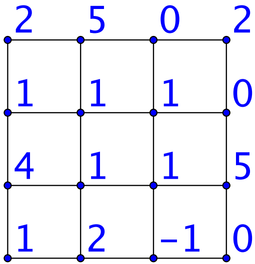
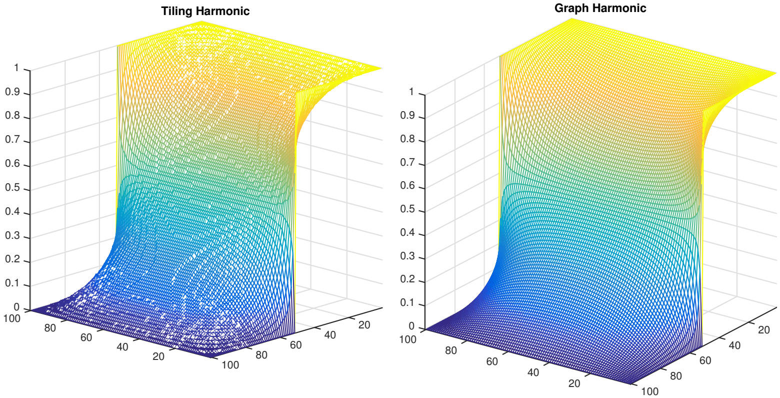
Results — Oscillating Boundary Values

Theorem

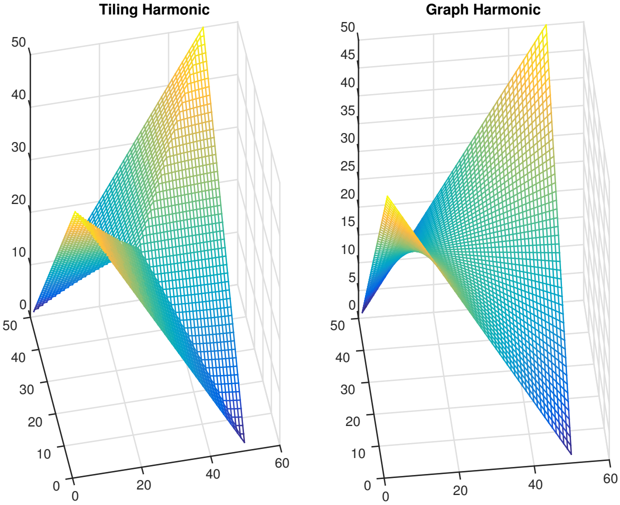
For every boundary square, consider the range of the boundary values on that square. If the intersection of these ranges is nonempty, then the only tiling-harmonic functions with these boundary values are constant on the interior.



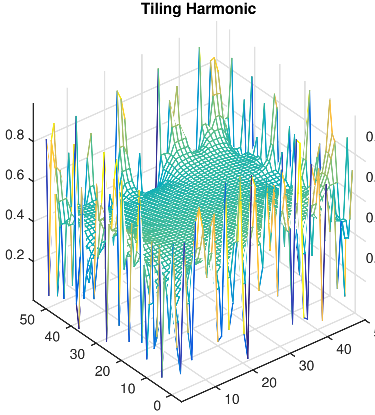
Tiling vs Graph Harmonic — Similarities

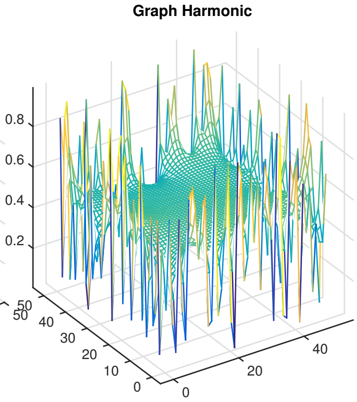


Tiling vs Graph Harmonic — Differences



Tiling vs Graph Harmonic — Random Boundary





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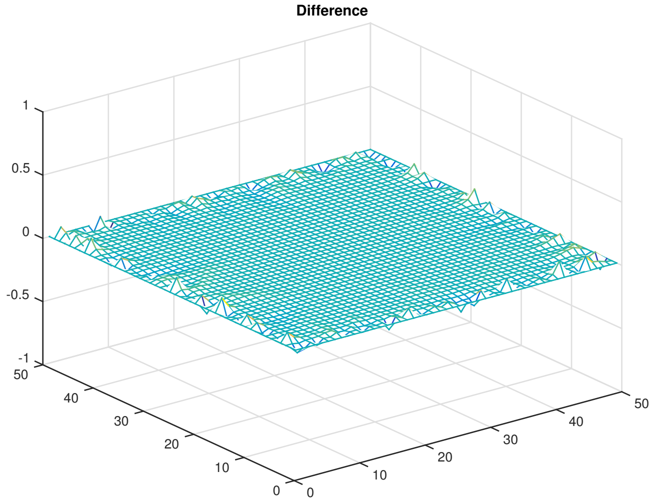

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- and My Parents.